Urban Land Use Theory

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0. INTRODUCTION

The modern theory of urban land use is essentially a revival of von Thünen's theory [147] of agricultural land use. Despite its significance as a monumental contribution to scientific thought, von Thünen's theory has languished without attracting the widespread attention of economists for over a century.[2] During that time human settlements grew extensively and outpaced the traditional guidance of urban design. It was the outburst of urban problems since the late 1950's that manifested the urgent need for a systematic theory of urban space and brought back the attention of location theorists and economists to von Thünen's theory.

Isard [68, Ch. 8] first suggested that von Thünen's theory could be extended in an urban context. Following the pioneering works of Beckmann [19] and Wingo [154], Alonso [2] succeeded in generalizing the concept of *bid rent curves*, the essence of von Thünen's theory, in an urban context. Since then, the subject has advanced considerably, inspiring a great deal of theoretical and empirical work. These efforts have culminated in Muth [93], Mills [81, 82],

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[2] For an excellent appraisal of Thünen's achievements from the viewpoint of modern economic theory, see Samuelson [123]. Samuelson states that "Thünen belongs to the Pantheon with Leon Walras, John Stuart Mill, and Adam Smith" (Samuelson [123], p. 1482]).

Henderson [61], Kanemoto [70] and Miyao [89], to name a few. Today, the theory of urban land use represents one of the youngest and liveliest fields in economics and regional science. This paper examines the state of the art in the economic theory of urban land use, including both positive and normative aspects of the theory. In most western societies, land is allocated among alternative uses mainly in private markets, with more or less public regulations. Many studies have revealed that strong regularities exist in the spatial structure of different urban areas. Positive theory provides explanations for these regularities. The existence of regularities, however, does not necessarily imply that the spatial structure of a city is a desirable one. It is the task of normative theory to identify the efficient spatial structure, and to suggest the means for achieving it.

The theory of urban land use is an especially appealing topic of study because traditional economic theory can not be readily applied. For although traditional theory aptly describes the competitive land market typical of most Western societies, it was designed to deal with spaceless problems. With land use, such considerations as spatial contiguity, externalities and durability of buildings become of vital importance. First, empirically one generally finds that households, as well as many firms and government agencies, choose one and only one location. In the terminology of traditional economic theory, this implies that there is a strong nonconvexity in consumption and production sets. Secondly, since the essence of cities is the presence of many people and firms in a limited area, externalities are a common feature. Public services, noise, pollution, and traffic congestion all involve externalities. Moreover, the necessity of non-price interactions such as information exchanges through face-to-face communication is one of the major reasons for people and firms to locate in a city. Finally, buildings and other urban infrastructure are among the most durable objects we make, and this limits the usefulness of the static theory. Because many spatial phenomena such as urban sprawl and renewal can be satisfactorily treated only in a dynamic framework, we eventually need to combine urban land use theory with capital theory. Clearly, the city is a fertile ground for economic study.

The first three sections of this paper concern the static theory of urban land use. Section 1 presents the basic theory of residential

land use within the context of the monocentric city in the absence of externalities. This section also introduces the main concepts and approach in the urban land use theory. Section 2 extends the basic theory by introducing public goods, neighborhood externalities and transport congestion. These two sections assume the location of firms is exogenously given, and are only concerned with the determination of household location. In Section 3, those models which simultaneously determine the location of households and firms are discussed. Finally, in Section 4, dynamic models of urban land use which explicitly consider the durability and adjustment cost of urban infrastructure are presented.

This paper aims not only to review the past works but also to present them in a coherent unified framework. For this purpose, we adopt the *bid rent function approach* which was introduced into an agricultural land use model by von Thünen [147], and later extended into the urban context by Alonso [2]. This approach is essentially the same as the indirect utility function approach which was introduced into an urban land use model by Solow [136]. A bid rent function transforms indifference curves in commodity space into indifference curves in urban space, i.e., bid rent curves. It is with these indifference curves defined in urban space that we will be able to graphically analyze the locational choice of the household (or firm). Moreover, since bid rent curves are stated as a pecuniary bid per unit of land, they are comparable among different land users. We will therefore be able to analyze competition for land among different agents, again, graphically in urban space. Furthermore, this bid rent function approach will enable us to conduct global analysis in contrast to the traditional local analysis via differential calculus. The same approach will be used in dynamic models of Section 4 by replacing the bid rent function with the bid (asset) price function of land.

1. BASIC THEORY OF RESIDENTIAL LAND USE

The simple models developed in this section serve as bases or building blocks for developing more complex models in the later sections. The models we will be dealing with are based on the following set of simplifying assumptions.

The city is monocentric. That is, it has a single, prespecified center of fixed size called the central business district (CBD). All job opportunities are located in the CBD. The transport system in the city is radial and dense in every direction; it is also free of congestion. The only travel is commuting of workers between residences and work places in the CBD. Travel within the CBD is ignored. The city is on a featureless plane, where all land parcels are identical, ready for residential use without further improvement. No local public goods or bads are in evidence, nor are there any neighborhood externalities.

The only spatial characteristic of each location in the city that matters to households is its distance from the CBD. Thus, the urban space can be treated as if it were one-dimensional. This is important since it greatly simplifies the analysis.

The market models presented in this section are largely due to Alonso [2] and Muth [93], the optimal models to Herbert and Stevens [62] and works by new urban economists in the early 1970s.[3]

1.1. Locational choice of the household

Imagine a household which arrives in a city and wishes to select a residence. As is typical in the economic analysis of consumer behavior, we assume that the household will maximize its utility subject to a budget constraint. We specify the utility function, $U(z, s)$, where z represents the amount of composite consumer good, and s the consumption of land, or the *lot size* of the house. The composite good is chosen as the numeraire, so its price is one. The household earns a fixed income Y per unit time which is spent on the composite good, land, and transportation. If the household is located at distance r from the CBD the budget constraint is given by $z + R(r)s = Y - T(r)$, where $R(r)$ is the unit land rent at r, and $T(r)$ the transport cost at r. Thus, we can express the residential

[3] In the early 1970's, numerous papers were written making use of optimal control or programming theory to analyze optimum or market equilibrium land use. This new approach was dubbed the "new urban economics" by Mills and Mackinnon [85]. Some of early important contributions are Dixit [33], Mills [81], Mirrlees [87], Oron, Pines and Sheshinski [102], Solow and Vickrey [137] and Solow [134, 135, 136]. For survey of the new urban economics, see Mills and MacKinnon [85]. Anas and Dendrinos [8], and Richardson [112].

choice of the household as

$$\max_{r,z,s} U(z, s), \quad \text{subject to } z + R(r)s = Y - T(r), \tag{1.1}$$

which is called the *basic model* of residential choice. This model is convenient in illustrating the basic mechanism of trade-off between accessibility and space in residential choice. In the subsequent analysis, we always assume that:

Assumption 1.1 (*well behaved utility function*). The utility function is differentiable, strictly quasi-concave and strictly increasing, and indifference curves do not cut the axes.

Assumption 1.2 (*increasing transport cost*) Marginal transport cost $T'(r) \equiv dT(r)/dr$ is always positive, $T(\infty) = \infty$ and $T(0) < Y$.

Assumption 1.3 (*normality of land*) Income effect on the ordinary demand for land is positive.

By directly solving the optimization problem implied by the basic model (1.1), we could ascertain, in a straightforward manner, the household's residential decision. But there is another approach, conceptually much richer, which leads to a desirable elaboration of theory. This approach, which mimics the von Thünen model of agricultural land use, requires the introduction of a concept called bid rent: *Bid rent* $\Psi(r, u)$ *is the maximum rent per unit land the household would be able to pay for residing at distance* r *while enjoying a fixed level of utility* u. Formally, bid rent may be defined as

$$\Psi(r, u) = \max_{z,s} \left\{ \frac{Y - T(r) - z}{s} \mid U(z, s) = u \right\}. \tag{1.2}$$

Or, we may first solve the utility constraint, $U(z, s) = u$, for z, and obtain the equation of the indifference curve as $z = Z(s, u)$. Then, the bid rent function can be redefined as

$$\Psi(r, u) = \max_{s} \frac{Y - T(r) - Z(s, u)}{s}, \tag{1.3}$$

which is an unconstrained maximization problem.[4] When we solve

[4] Note that because of Assumption 1.1, whenever it exists, the optimal s for the maximization problem of (1.3) is positive. When there is no solution for this maximization problem, we define $\Psi(r, u) = 0$ and $S(r, u) = \infty$.

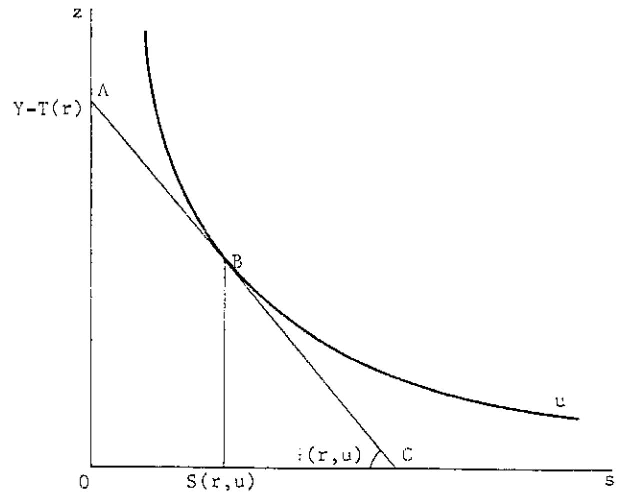

FIGURE 1.1 Bid rent $\Psi(r, u)$ and bid-max lot size $S(r, u)$.

the maximization problem of (1.2) or (1.3), we obtain the optimal lot size, $S(r, u)$, which is called the *bid-max lot size.* In order to emphasize that the basic parameters of bid rent $\Psi(r, u)$ and bid-max lot size $S(r, u)$ are net income $Y - T(r)$ and utility level u, we often express them as

$$\Psi(r, u) \equiv \psi(Y - T(r), u), \qquad S(r, u) \equiv s(Y - T(r), u). \quad (1.4)$$

Graphically, as depicted in Figure 1.1, bid rent $\Psi(r, u)$ is given by the slope of the budget line at distance r which is just tangent to indifference curve u;[5] bid-max lot size $S(r, u)$ is determined from the tangency point B.

EXAMPLE 1 In the case of a log-linear utility function,

$$U(z, s) = \alpha \log z + \beta \log s, \quad \text{where} \quad \alpha > 0, \beta > 0, \alpha + \beta = 1,$$

we have $Z(s, u) = s^{-\beta/\alpha} e^{u/\alpha}$. Solving the maximization problem of (1.3) with this utility function, we have

$$\Psi(r, u) = \alpha^{\alpha/\beta} \beta (Y - T(r))^{1/\beta} e^{-u/\beta}. \quad (1.5)$$

$$S(r, u) = \beta(Y - T(r))/\Psi(r, u) = \alpha^{-\alpha/\beta}(Y - T(r))^{-\alpha/\beta} e^{u/\beta}. \quad (1.6)$$

[5] More precisely, if we denote the angle ACO by θ, then $\Psi(r, u) = \tan \theta$. But, for simplicity, we use this graphical expression throughout the paper.

In order to relate bid rent $\Psi(r, u)$ and bid-max lot size $S(r, u)$ to familiar microeconomic notions, we may appeal, once again, to Figure 1.1. In Figure 1.1 we may interprete that indifference curve u is tangent to budget line AC from above at point B. This implies the following: if we define the *ordinary demand*, $\hat{s}(I, R)$, for land from the solution of the next utility maximization problem,

$$\max_{z,s} U(z, s), \text{ subject to } z + Rs = I,$$

then it holds identically that

$$S(r, u) \equiv \hat{s}(Y - T(r), \Psi(r, u)). \quad (1.7)$$

Or, we may interpret Figure 1.1 as budget line AC being tangent to indifference curve u from below at B. This implies that: if we define the *compensated demand*, $\bar{s}(R, u)$, for land from the solution of the next expenditure minimization problem,

$$\min_{z,s} z + Rs, \quad \text{subject to } U(z, s) = u,$$

then it holds identically that

$$S(r, u) \equiv \bar{s}(\Psi(r, u), u). \quad (1.8)$$

Identities (1.7) and (1.8) turn out to be very useful in deriving qualitative results.[6]

Next, we examine important properties of bid rent and bid-max lot size functions. Through an application of the envelope theorem to (1.3), we have

$$\frac{\partial \Psi(r, u)}{\partial r} = -\frac{T'(r)}{S(r, u)} < 0. \quad (1.9)$$

Hence, from identity (1.8),

$$\frac{\partial S(r, u)}{\partial r} = \frac{\partial \bar{s}}{\partial R} \frac{\partial \Psi(r, u)}{\partial r} > 0, \quad (1.10)$$

[6] We can also derive the following identities. Define the *indirect utility function*, $V(I, R) = \max_{z,s} \{U(z, s) \mid z + Rs = I\}$; define the expenditure function, $E(R, u) = \min_{z,s} \{z + Rs \mid U(z, s) = u\}$. And, put $I(r) = Y - T(r)$. Then, we can see from Figure 1.1 that $V(I(r), \psi(I(r), u)) \equiv u$ and $E(\psi(I(r), u), u) \equiv I(r)$. That is, ψ and V are inverse to each other as are ψ and E. In fact, in Solow [136], bid rent function ψ has been defined by solving the relation, $V(I, R) = u$, for R.

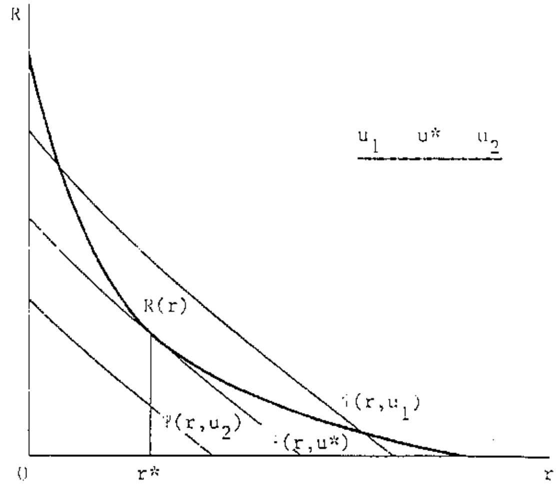

FIGURE 1.2 Determination of the equilibrium location.

which is positive since $\partial \hat{s}/\partial R$ is always negative. Similarly,

$$\frac{\partial \Psi(r, u)}{\partial u} = -\frac{1}{S(r, u)} \frac{\partial Z(s, u)}{\partial u} < 0, \tag{1.11}$$

which is negative since $\partial Z(s, u)/\partial u$ is positive from Assumption 1.1. So, from identify (1.7),

$$\frac{\partial S(r, u)}{\partial u} = \frac{\partial \hat{s}}{\partial R} \frac{\partial \Psi(r, u)}{\partial u} > 0, \tag{1.12}$$

which is positive since $\partial \hat{s}/\partial R$ is negative from Assumption 1.3. Therefore, we can conclude that *bid rent $\Psi(r, u)$ is decreasing in both r and u, and bid-max lot size $S(r, u)$ is increasing in both r and u.*

We are now ready to explain how the equilibrium (or optimal)

If we further assume that transport cost is linear or concave in r (i.e., $T''(r) \leqq 0$), then from (1.9) and (1.10),

$$\frac{\partial^2 \Psi(r, u)}{\partial r^2} = -\frac{T''(r)}{S(r, u)} + \frac{T'(r)}{S(r, u)} \frac{\partial S(r, u)}{\partial r} > 0,$$

which means bid rent curves are strictly convex in r.

location of the household is determined, given the basic model (1.1) and the land rent curve $R(r)$ of the city. In Figure 1.2, the market rent curve $R(r)$ is depicted; and a set of bid rent curves are superimposed on it. It is evident from the figure that *the equilibrium location of the household is distance r^* at which a bid rent curve $\Psi(r, u^*)$ is tangent to the market rent curve $R(r)$ from below.* That is, when the household decides to locate somewhere in the city, it is obliged to pay the market land rent. At the same time, the household will maximize its utility. Since the utility of bid rent curves increases towards the origin, the highest utility will be achieved at a location at which a bid rent curve is tangent to the market rent curve from below. This conclusion can be formally stated as follows:

Rule 1.1. Given the market rent curve $R(r)$, u^* is the equilibrium utility level of the household, and r^* is an optimal location if and only if $R(r^*) = \Psi(r^*, u^*)$ and $R(r) \geqq \Psi(r, u^*)$ for all r.

Given that curves $R(r)$ and $\Psi(r, u^*)$ are smooth at r^*, the above rule implies

$$\frac{\partial \Psi(r^*, u^*)}{\partial r} = R'(r^*). \tag{1.13}$$

Thus, recalling equation (1.9), we have

$$T'(r^*) = -R'(r^*)S(r^*, u^*) \tag{1.14}$$

This result, called *Muth's Condition,* asserts that at the equilibrium location the marginal transport cost $T'(r)$ equals the marginal land cost saving, $-R'(r)\, S(r, u^*)$.

Next, we can study the relative locations of different households having different bid rent functions. A general rule for ordering equilibrium locations of different households with respect to the distance from the CBD is as follows:

Rule 1.2. If the equilibrium bid rent curve $\Psi_i(r, u_i^*)$ of household i and equilibrium bid rent curve $\Psi_j(r, u_j^*)$ of household j intersect only once and if $\Psi_i(r, u_i^*)$ is steeper than $\Psi_j(r, u_j^*)$ at the intersection, then the equilibrium location of household i is closer to the CBD than that of household j.

This result is depicted in Figure 1.3. In order to apply this rule, however, we must know beforehand which equilibrium curve is

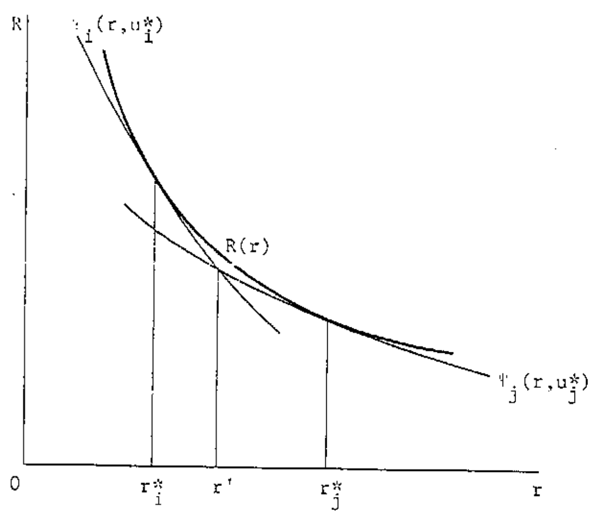

FIGURE 1.3 Ordering of equilibrium locations.

steeper at the intersection. In general, this information is difficult to obtain *a priori*. Matters can be greatly simplified, however, if we are able to determine the *relative steepness of bid rent functions*. We say that bid rent function Ψ_i is steeper than bid rent function Ψ_j if and only if *at every intersection of each pair of bid rent curves*, apiece for household i and j, the former is always steeper than the latter.[8] Figure 1.4 gives an illustration. From Rule 1.2 combined with this definition, we can state:

Rule 1.3. If the bid rent function of household i is steeper than that of household j, then the equilibrium location of household i is closer to the CBD than that of household j.

Although the applicability of this rule is limited, it becomes very useful in comparative static analysis, where the effects of difference in model parameter values are examined. In fact, almost always when a definite conclusion can be obtained from a comparative static analysis of household location, the relative steepness of bid

[8] Stated formally, function Ψ_i is steeper than function Ψ_j if and only if the following condition is met: whenever $\Psi_i(r, u_i) = \Psi_j(r, u_j) > 0$, then either $-\partial\Psi_i(r, u_i)/\partial r > -\partial\Psi_j(r, u_j)/\partial r$, or $\partial\Psi_i(r, u_i)/\partial r = \partial\Psi_j(r, u_j)/\partial r$ and $\partial^2\Psi_i(r, u_i)/\partial r^2 < \partial^2\Psi_j(r, u_j)/\partial r^2$.

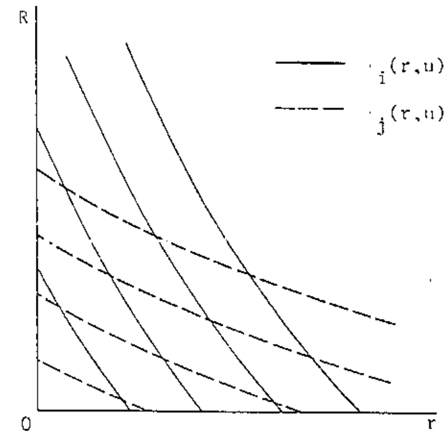

FIGURE 1.4 Relative steepness of bid rent functions.

rent functions (determined by parameter values) can be ascertained. An important example is the effect of income level on household location.[9]

In the context of basic model (1.1), let us arbitrarily specify two income levels, $Y_1 < Y_2$. It is assumed that both households possess the same utility function and face the same transport cost function. Denote by $\Psi_i(r, u)$ and $S_i(r, u)$ the bid rent and bid-max lot size of the household with income Y_i $(i = 1, 2)$. Let us arbitrarily take a pair of bid rent curves, $\Psi_1(r, u_1)$ and $\Psi_2(r, u_2)$, and suppose that they intersect at some distance $r' : \Psi_1(r', u_1) = \Psi_2(r', u_2) \equiv \bar{R}$. Recall indentity (1.7). Since $Y_1 - T(r') < Y_2 - T(r')$, from the normality of land,

$$S_1(r', u_1) = \hat{s}(Y_1 - T(r'), \bar{R}) < \hat{s}(Y_2 - T(r'), \bar{R}) = S_2(r', u_2).$$

Thus, from (1.9), $-\partial\Psi_1(r', u_1)/\partial r > -\partial\Psi_2(r', u_2)/r$. Since we have arbitrarily chosen two bid rent curves, this result means that function Ψ_1 is steeper than Ψ_2. Thus, from Rule 1.3, we may state the following proposition: *Other aspects being equal, higher income households locate further from the CBD than lower income house-*

[9] For other examples, see Sections 1.4.A and 1.4.B.

holds. This result has been often used to explain the residential pattern observed in the United States.[10]

1.2. Equilibrium land use

We next examine equilibrium configurations of residential land as determined through competitive land markets. Following Wheaton [148], we may classify market models as *closed city models* and *open city models*. In the closed city model, the population of the city is exogenously fixed, while equilibrium utility is determined endogenously. In the open city model, households are assumed to be able to move costlessly across the city boundary; hence, the utility of residents equals that of the rest of the economy which is exogeneously fixed, while the population of the city is determined endogenously. The closed city model is a useful conceptual device when analyzing urban land use in large cities or "average cities" of developed countries. On the other hand, the open city model better describes urban conditions in developing countries which have surplus labor in rural areas. In the latter case, rural life often establishes the base utility level of the economy. The emphasis here will be on the closed city model, because it is more fundamental from a theoretical point of view. In both models, we must also specify the form of land ownership. Two popular specifications are the *absentee ownership model*, in which land is owned by absentee landlords, and the *public ownership model*, in which the revenue from land is equally shared among city residents.

This subsection will examine the four cases in turn. For simplicity, let us assume that all households in the economy are identical.[11] We denote by $L(r)$ the amount of land available for

[10] Recall, however, that this result depends critically on the assumption that transport costs are independent of income. A completely reversed spatial pattern can be observed in many European, Latin American, and Asian cities. In the United States as well, luxury apartments and townhouses are often found near the urban center. See Alonso [2, Ch. 6], Muth [93] and Wheaton [150] for empirical studies on household location. These observations suggest that factors other than income including value of time, family structure, externalities, and dynamic factors also affect residential choices, and spatial patterns. These factors will be introduced one by one in the rest of the paper.

[11] The equilibrium pattern of residential land with multiple income classes was studied by Muth [93], Solow [136], Hartwick, Schweizer and Varaiya [56], Ando [10] and Fujita [45, 46]. Beckmann [20], with some corrections by Montesano [91], considered the case of a continuous, Pareto income distribution. See Miyao [88] for a dynamic stability analysis of boundaries between different income classes.

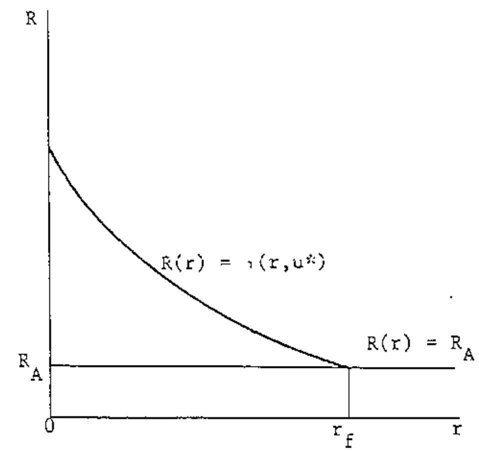

FIGURE 1.5 Equilibrium spatial configuration.

residential use at each distance r, which is assumed to be positive for all $r > 0$. It is also assumed that land not occupied by households is used for agriculture, yielding a constant rent R_A.

Case 1 is the closed city model under absentee land ownership. There are M identical households in the city. We assume that households behave according to the basic model (1.1), and that the household income Y is given exogenously.[12] Then, the bid rent function $\Psi(r, u)$ and bid-max lot size function $S(r, u)$ can be derived as explained before. Let u^*, $R(r)$, r_f and $n(r)$ be the utility level, land rent curve, urban fringe distance, and household distribution (i.e., the number of households between r and $r + dr$ equals $n(r)dr$) in equilibrium, respectively. From Rule 1.1, the equilibrium land rent $R(r)$ must be equal to the equilibrium bid rent $\Psi(r, u^*)$ everywhere in the residential area. Thus, we have

$$R(r) = \Psi(r, u^*) \quad \text{for} \quad r \leqq r_f, \qquad = R_A \quad \text{for} \quad r \geqq r_f. \tag{1.15}$$

This relationship is described in Figure 1.5. The household distribution is given by

$$n(r) = L(r)/S(r, u^*) \quad \text{for} \quad r < r_f. \tag{1.16}$$

[12] For endogenous determination of wage income Y, see, for example, Solow [136], Henderson [61] and Kanemoto [70].

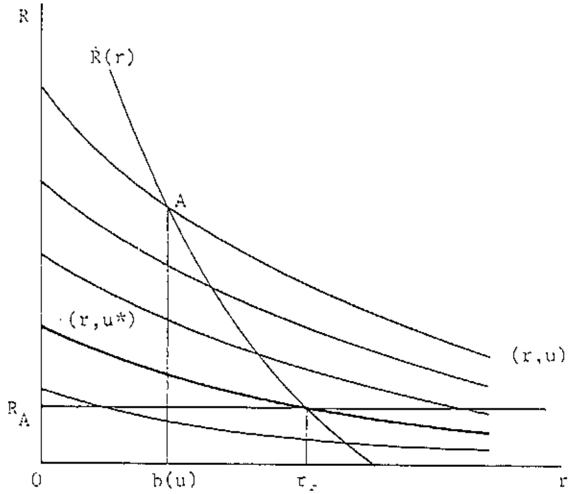

FIGURE 1.6 Boundary rent curve $\bar{R}(r)$ for equilibrium land use.

Hence, the population constraint is given by

$$\int_0^{r_f} \frac{L(r)}{S(r, u^*)} dr = M. \tag{1.17}$$

Two unknowns u^* and r_f can be determined from (1.17) and the next boundary rent condition,

$$\Psi(r_f, u^*) = R_A. \tag{1.18}$$

The existence and uniqueness of the equilibrium solution can be demonstrated by using the concept of the *boundary rent curve.*[13] Under each value of u, solve the next equation for b,

$$\int_0^{b} \frac{L(r)}{S(r, u)} dr = M, \tag{1.19}$$

and obtain the *outer boundary function*, $b(u)$, of residential area. For each given u, $b(u)$ marks a distance on the corresponding bid rent curve $\Psi(r, u)$, such as point A in Figure 1.6. By changing u, we

[13] A similar concept was introduced by Kanemoto [70, Ch. 6], in his stability analysis of mixed cities. Fujita [45, 46] generalized it to the case of variable lot size, and used it for the study of the existence and uniqueness of the solution.

can obtain a curve, $\bar{R}(r)$, called the *boundary rent curve*, as depicted in Figure 1.6. If the inverse of $r = b(u)$ is denoted by $U(r)$, the boundary rent curve may be defined as

$$\bar{R}(r) \equiv \Psi(r, U(r)). \tag{1.20}$$

By definition, $\bar{R}(r)$ represents the market land rent at r when the residential boundary occurs at r. Suppose we have obtained the curve $\bar{R}(r)$. Then, as depicted in Figure 1.6, the equilibrium fringe distance r_f is given by the point where $\bar{R}(r_f) = R_A$. Then, equilibrium utility level u^* can be obtained from the relationship, $\Psi(r_f, u^*) = R_A$. Under Assumptions 1.1 to 1.3, it is not difficult to show that the boundary rent curve $\bar{R}(r)$ is continuously decreasing in r, becomes zero as r approaches a certain finite distance, and becomes infinitely high as r approaches zero (see Fujita [45]). Therefore, as demonstrated in Figure 1.6, we can conclude that *there exists a unique equilibrium for the closed city model under absentee land ownership.*[14]

For case 2, the open city model under absentee land ownership, the equilibrium is trivially simple to obtain. If the national utility level is given by a constant u^*, then the urban fringe distance r_f can be obtained from relation (1.18). The equilibrium population M can be determined by (1.17), and the land rent curve $R(r)$ is given by (1.15).

Next, for Case 3 let us consider the closed city model under public land ownership, which was introduced by Solow [136]. The city residents form a government, which rents the land for the city from rural landlords at agricultural rent R_A. The city government in turn, subleases the land to city residents at the competitively determined rent, $R(r)$, at each location. Define the *total differential rent* (TDR) from the city by

$$\mathrm{TDR} = \int_0^{r_f} (R(r) - R_A) L(t)\, dr. \tag{1.21}$$

Assume that there are M identical households in the city. Then, the income of each household is its wage income (or non-rent income)

[14] See Fujita [45, 46] for the extension of the boundary rent curve approach for the proof of existence and uniqueness of equilibrium in the case of multiple household types.

Y plus a share of land rent, TDR/M. Thus, the residential choice behavior of each household can be formulated as follows:

$$\max_{r,z,s} U(z, s), \text{ subject to } z + R(r)s = Y + (\text{TDR/M}) - T(r). \tag{1.22}$$

Since TDR is yet unknown, we cannot apply the boundary rent curve approach in order to examine the existence and uniqueness of the equilibrium. We will prove them in the next subsection via the existence and uniqueness of the optimal land use.

Case 4 is the open city model under public land ownership. As in Case 3, each resident of the city gets a wage income Y plus a share of land rent, TDR/M. The residential choice behavior of each household can be described, as before, by (1.22). However, both TDR and M are now unknown; the utility of residents is fixed at the national utility level u. We will prove the existence and uniqueness of the equilibrium in the next subsection.

Comparative statics were studied by Wheaton [148] in the case of single household type; and by Wheaton [149], Hartwick, Schweizer and Varaiya [56], and Arnott, MacKinnon and Wheaton [14] in the case of multiple income classes. Take the case of single household type in the context of the closed city model under absentee land ownership, where the residential choice of each household is represented by the basic model (1.1). Assume for simplicity that $T(r) = ar$. Then, four parameters are agricultural rent R_A, population M, marginal transport cost a, and income Y. Under Assumptions 1.1, 1.2 and 1.3, either by traditional differential calculus (Wheaton [148], Hartwick, *et al.* [56]), or by the boundary rent curve approach (Fujita [46]), we can obtain the following results:

$$\frac{du^*}{dR_A} < 0, \quad \frac{dr_f}{dR_A} < 0, \quad \frac{dR(r)}{dR_A} > 0 \quad \text{and} \quad \frac{dS(r, u^*)}{dR_A} < 0 \quad \text{for all } r, \tag{1.23}$$

$$\frac{du^*}{dN} < 0, \quad \frac{dr_f}{dN} > 0, \quad \frac{dR(r)}{dN} > 0 \quad \text{and} \quad \frac{dS(r, u^*)}{dN} < 0 \quad \text{for all } r, \tag{1.24}$$

$$\frac{du^*}{da} < 0, \quad \frac{dr_f}{da} < 0, \quad \frac{dR(0)}{da} > 0, \quad \frac{dS(0, u^*)}{da} < 0, \tag{1.25}$$

$$\frac{du^*}{dY} > 0, \quad \frac{dr_f}{dY} > 0, \quad \frac{dS(0, u^*)}{dY} > 0. \tag{1.26}$$

These results are hardly surprising. On the other hand, the effect of income change on the land rent at the city center depends crucially on the distribution of land, and we can show that:

$$\frac{dR(0)}{dY} \gtreqless 0 \quad \text{as} \quad L'(r) \lesseqgtr 0 \quad \text{for all } r. \tag{1.27}$$

Since $L'(r)$ is positive in most cities, from (1.25) and (1.27) we can conclude that both a decrease in marginal transport cost and an increase in income will make the land rent curve flatter (i.e., $dR(0) < 0$ and $dr_f > 0$), and will also make the population density curve flatter (i.e., $dS(0, u^*) > 0$ and $dr_f > 0$). This result describes the suburbanization trend observed in the United States and other developed countries (at least, prior to the oil shocks at the beginning of 1970s).[15]

Finally, the following relationship between TDR (total differential rent) and TTC (total transport cost), which was obtained by Arnott [12], is noteworthy because of its simplicity and generality. In the context of the basic model of (1.1), suppose that transport cost function is linear (i.e., $T(r) = ar$), and that distribution of land is given by $L(r) = \theta r^{\lambda}$ where $\lambda > -1$. Then, for all four cases, a simple calculation using relation (1.14) yields that[16]

$$\text{TDR} = \text{TTC}/(\lambda + 1). \tag{1.28}$$

Hence, in the case of linear city ($\lambda = 0$), TDR = TTC; and in the case of a circular or fan-shaped city ($\lambda = 1$), TDR = TTC/2. It is not difficult to see from the derivation that as long as *each* household

[15] For empirical studies on historical trend in population density gradients, see, for example, Mills [82, 83] and Parr and Jones [107].

[16] $\text{TTC} = \int_0^{r_f} T(r)n(r)\,dr = \int_0^{r_f} ar\theta r^{\lambda}/S(r, u^*)\,dr = \int_0^{r_f} -R'(r)\theta r^{\lambda+1}\,dr$ (from (1.14)) $= -R_A\theta r_f^{\lambda+1} + \int_0^{r_f} R(r)(\lambda+1)\theta r^{\lambda}\,dr$ (from integration by parts) $= (\lambda+1)\int_0^{r_f}(R(r) - R_A)\theta r^{\lambda}\,dr = (\lambda+1)\text{TDR}$. In the case of fixed lot size, this relationship was obtained by Mohring [90] from ingeneous geometric reasoning.

has a *directionally linear* transport cost function, relationship (1.28) holds for both equilibrium cities and optimal cities.[17]

1.3. Optimal land use, optimal vs. equilibrium

This subsection studies the optimal allocation of residential land and households in a city, and examines the relationship between the optimal land use and equilibrium land use; in particular, the ability of competitive market to sustain an optimal allocation of residential land and households. Exactly what optimal land use is, of course, depends on how the objective function is specified. In spaceless economics, it is common to maximize a Benthamite social welfare function, which is the sum of utilities of individual households (an unweighted sum in the case of identical households). However, this is not the most convenient approach for the land use problem, because maximization of a Benthamite welfare function results in the assignment of different utility levels to identical households depending on their locations. Such a result is referred to as *Mirrlees' inequality* or *unequal treatment of equals*, and is a unique phenomenon due to the nonconvexity introduced by space.[18] Since competitive markets treat all equals equally, it is clear that the maximization of a Benthamite welfare function is not the most direct approach for investigating the efficiency of land markets. It turns out that the so called *Herbert-Stevens model* (*HS model*) is a convenient formulation of optimization problems for land use theory. In this model, the objective is to maximize net revenue subject to a set of prespecified target utility levels for all household types. The model is designed so that its solution is always efficient

[17] That is, different households may have different linear transport cost functions of which marginal transport costs may depend on the direction from the city center. If the time cost is involved in transport, we must also include the time cost in calculating TTC. It is not difficult to see from the derivation that relation (1.28) holds both in equilibrium cities and optimal cities. Moreover, if land rent $R(r)$ is replaced with land price $P(r)$ (i.e., asset price of land), the same relationship holds between the total differential land price and the total discounted value of transport costs for dynamic cities in Section 4 (under perfect foresight without tulip-mania expectation).

[18] This phenomenon was discovered by Mirrlees [87]. For the explanation of this phenomenon and for further discussion of this subject, see Riley [114, 115], Arnott and Riley [15], Levhari, Oron and Pines [78], and Kanemoto [70, Appendix I].

(i.e., Pareto-optimal), and all efficient allocations can be obtained by simply varying the target utility levels.

As before, we have a monocentric city with M identical households.[19] The utility function of each household is given by $U(z, s)$, and transport cost by $T(r)$. Suppose we arbitrarily choose a target utility level, u, and require that all households shall attain this. Next, suppose we choose a household distribution $n(r)$, a lot size function $s(r)$, and a residential fringe distance r_f which together satisfy the following land and population constraints:

$$s(r)n(r) \leqq L(r) \quad \text{at each} \quad r \leqq r_f, \tag{1.29}$$

$$\int_0^{r_f} n(r)\, dr = M. \tag{1.30}$$

Then, the *total cost* C for achieving target utility u can be calculated as follows:

$$C = \text{transport costs} + \text{composite good costs} + \text{opportunity land costs}$$
$$= \int_0^{r_f} (T(r) + Z(s(r), u) + R_A s(r))n(r)\, dr, \tag{1.31}$$

where $Z(s, u)$ is the inverse of $u = U(z, s)$ for z. The problem is to choose an allocation, $(n(r), s(r), r_f)$, that minimizes the total cost C subject to land constraint (1.29) and population constraint (1.30). This problem can be more conveniently expressed in terms of net revenue. If we assume that the per capita income of the city is given by Y, then the total income of the city is MY. Let Y be some fixed number determined independently of the residential land use pattern. The *net revenue* NR from allocation $(n(r), s(r), r_f)$ is

$$\text{NR} = MY - C$$
$$= \int_0^{r_f} (Y - T(r) - Z(s(r), u) - R_A s(r))n(r)\, dr. \tag{1.32}$$

Since MY is assumed to be a constant, minimization of C is equivalent to maximization of NR, and the *Herbert-Stevens*

[19] For the similar discussion of optimal land use with multiple household types, see Ando [10] and Fujita [45, 46].

problem, HS(u), can be stated as follows:[20]

$$\max_{n(r),s(r),r_f} \text{NR} = \int_0^{r_f} (Y - T(r) - Z(s(r), u) - R_A s(r))n(r)\, dr,$$

subject to constraints (1.29) and (1.30).

Considering that NR represents a benefit (or cost, if negative) for the rest of the economy, it is obvious that the solution of any HS-problem is socially efficient, and any efficient allocation under the equal utility condition is a solution of some HS-problem.

Now to state the optimality conditions for the above HS-problem, let us define the *bid rent function with income subsidy Q* as follows:

$$\Psi(r, u, Q) = \max_s \frac{Y + Q - T(r) - Z(s, u)}{s}, \tag{1.33}$$

and denote the corresponding bid-max lot size by $S(r, u, Q)$.[21] Suppose Assumptions 1.1 and 1.2 hold. Then, for an allocation $(n(r), s(r), r_f)$ to be optimal, it is necessary and sufficient that there exist multipliers $R(r)$ and $\hat{Q}$ such that:[22]

$$R(r) = \Psi(r, u, \hat{Q}) \quad \text{for} \quad r \leqq r_f, \quad = R_A \quad \text{for} \quad r \geqq r_f, \tag{1.34}$$

$$s(r) = S(r, u, \hat{Q}) \quad \text{and} \quad n(r) = L(r)/S(r, u, \hat{Q}) \quad \text{for} \quad r < r_f, \tag{1.35}$$

$$\int_0^{r_f} L(r)/S(r, u, \hat{Q})\, dr = M. \tag{1.36}$$

$R(r)$ can be interpreted as the shadow land rent at r, and $\hat{Q}$ the shadow income subsidy (or shadow cost) per household. The concept of income subsidy will arise again in comparing the optimal

[20] Since this is a continuous version of the problem introduced by Herbert and Stevens [62], we call it a Herbert–Stevens problem. A continuous version was introduced by Ando [10], Yang and Fujita [157], and Fujita [46]. An HS-problem is the dual to the maximization of the common utility level subject to the resource constraint of the city (i.e., a balanced budget). In fact, the latter problem formulation, introduced by Dixit [33] and Oron, Pines and Sheshinski [102], is more popular in the literature. However, the HS problem-formulation appears to be more convenient for comparison of market allocations and optimal allocations.

[21] Similarly to (1.4), we may express $\Psi(r, u, Q) \equiv \psi(Y + Q - T(r), u)$ and $S(r, u, Q) \equiv s(Y + Q - T(r), u)$.

[22] For the derivation of the following conditions, see Ando [10] or Fujita [46].

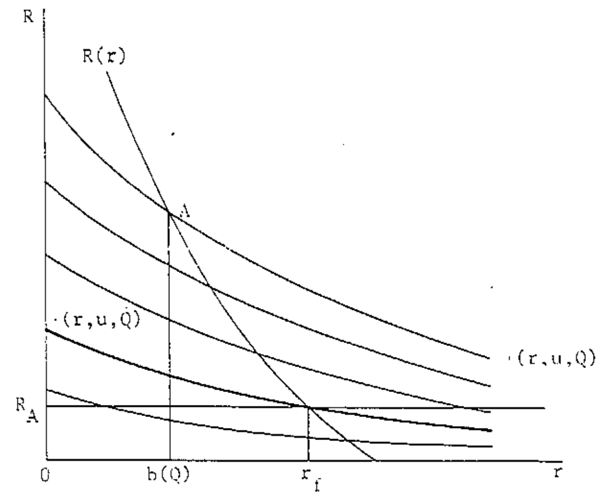

FIGURE 1.7 Boundary rent curve $\bar{R}(r)$ for optimal land use.

land use with the equilibrium land use. Briefly, such a subsidy is one way for government to achieve the optimal solution of the $HS(u)$-problem through a competitive market.

Recalling the boundary rent curve approach, we can determine two unknowns $\hat{Q}$ and r_f. Under each value of Q, we solve the next equation for b,

$$\int_0^b \frac{L(r)}{S(r, u, Q)}\, dr = M, \tag{1.37}$$

and obtain the outer boundary function, $b(Q)$. For each given Q, $b(Q)$ marks a distance on the corresponding bid rent curve $\Psi(r, u, Q)$, such as point A in Figure 1.7. By changing Q, we can obtain the boundary rent curve, $\bar{R}(r)$, as depicted in Figure 1.7. That is,

$$\bar{R}(r) \equiv \Psi(r, u, Q(r)), \tag{1.38}$$

where $Q(r)$ is the inverse of $r = b(Q)$. With $\bar{R}(r)$, then, as depicted in Figure 1.7, r_f is determined from the relation $\bar{R}(r_f) = R_A$, and $\hat{Q}$ from $\Psi(r_f, u, \hat{Q}) = R_A$.

In addition to Assumptions 1.1, 1.2 and 1.3, suppose the following assumption is satisfied.

Assumption 1.4 z and s are perfectly substitutable. That is, on each indifference curve $u = U(z, s)$, s approaches zero as z approaches infinity.[23]

Then, as before, $\bar{R}(r)$ is continuously decreasing in r, becomes zero as r approaches a certain finite distance, and $\lim_{r\to 0} \bar{R}(r) = \infty$. Therefore, as demonstrated in Figure 1.7, *for any given u the HS(u)-problem has a unique solution.* One can also show that if we donote by $\hat{Q}(u)$ the shadow income subsidy at the solution of each HS(u)-problem, then $\hat{Q}(u)$ is strictly increasing in u.

Next, let us examine the relationship between the optimal land use and equilibrium land use. Keeping the context of the basic model (1.1), let us introduce a new parameter, Q, which represents an *income subsidy* per household. Then, the residential choice behavior of each household is:

$$\max_{r,z,s} U(z, s), \quad \text{subject to } z + R(r)s = Y + Q - T(r). \quad (1.39)$$

For the closed city model with M identical households under absentee ownership, the equilibrium conditions are:

$$R(r) = \Psi(r, u^*, Q) \quad \text{for} \quad r \leqq r_f, \quad = R_A \quad \text{for} \quad r \geqq r_f, \quad (1.40)$$

$$n(r) = L(r)/S(r, u^*, Q) \quad \text{for} \quad r < r_f, \quad (1.41)$$

$$\int_0^{r_f} L(r)/S(r, u^*, Q)\, dr = M. \quad (1.42)$$

where functions $\Psi(r, u, Q)$ and $S(r, u, Q)$ have been defined from (1.33). We may call this market problem the *Alonso-Muth problem with income subsidy* Q, and denote it by AM(Q). As in subsection 1.2, we can see that for each Q such that $Y + Q > T(0)$, the AM(Q)-problem has a unique solution. If we denote the equilibrium utility of each AM(Q)-problem by $u^*(Q)$, then $u^*(Q)$ is strictly increasing in Q.

Comparing optimality conditions (1.34)–(1.36) with equilibrium conditions (1.40)–(1.42), we can see that functions $\hat{Q}(u)$ and $u^*(Q)$ are inverse to each other: $\hat{Q}(u^*(Q)) \equiv Q$ and $u^*(\hat{Q}(u)) \equiv u$. Hence, we can conclude that for any u, the $HS(u)$-problem and $AM(\hat{Q}(u))$-

[23] If we assume that the total amount of land, $\int_0^\infty L(r)\,dr$, is infinite, then this assumption is not necessary for the existence and uniqueness of the solution.

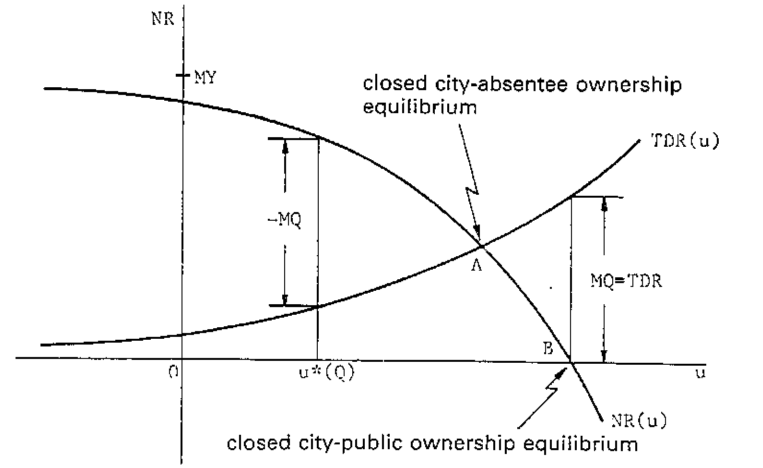

FIGURE 1.8 Relationship between optimal solutions and equilibrium solutions (M fixed).

problem have the same solution; similarly, for any Q such that $Y + Q > T(0)$, the AM(Q)-problem and HS($u^*(Q)$)-problem have the same solution. Therefore, the solution of any HS-problem can be obtained by using the AM-model and choosing an appropriate income subsidy; conversely, the solution of any AM-problem can be obtained by using the HS-model and choosing an appropriate target utility. The fact that the solution of any HS-problem is efficient, implies that the solution of any AM-problem is also efficient.[24]

The above result can be summarized graphically. Let NR, TDR and Q, respectively, be the value of the objective function, the total differential rent, and the shadow income subsidy at the solution of each HS-problem. Then, a simple calculation reveals that the following relationship holds (under any parameter values of u and M):

$$\text{NR} = \text{TDR} - MQ. \quad (1.43)$$

The relationship between the two curves NR and TDR is depicted in Figure 1.8. Since Q is increasing in u ($d\hat{Q}(u)/du > 0$), before

[24] This conclusion is, of course, hardly surprising in light of traditional welfare economic theory. Nevertheless, it is worth reconfirming in the context of the model with continuous space.

they intersect at point A the difference between the two curves is decreasing in u; after point A, the difference is increasing in u. Given an income subsidy (or tax) Q, the equilibrium utility $u^*(Q)$ at the solution of $AM(Q)$-problem can be obtained as in Figure 1.8. In particular, Q equals zero at point A. Hence, this point corresponds to the market equilibrium for the closed city model under absentee land ownership in subsection 1.2 (case 1). Next, at point B which exists uniquely since curve $\mathrm{NR}(u)$ is continuous and $\lim_{u\to\infty} \mathrm{NR}(u) = -\infty$, we have $\mathrm{NR} = 0$ and hence $Q = \mathrm{TDR}/M$. Hence, point B corresponds to the equilibrium of the public ownership market model of (1.22) under fixed population. Thus, we can conclude that *there exists a unique equilibrium for the closed city model under public land ownership in subsection* 1.2 (*case* 3).

Thus far, we have been concerned with the effect of change in parameter u on the solution of the HS-problem, while population M is kept constant. Next, let us examine the effect of change in parameter M on the solution of the HS-problem, while the target utility level is kept constant at u. As depicted in Figure 1.9, the net revenue curve, NR(M), is continuous and strictly concave in M, and $\lim_{M\to\infty} \mathrm{NR}(M) = -\infty$ (see Fujita [46]). Let $\hat{u}$ be the supreme utility level determined by the relation, $\Psi(0, \hat{u}, 0) = R_A$. We can see that if $u < \hat{u}$, then $\partial\mathrm{NR}/\partial M > 0$ at $M = 0$, and hence intersection B exists uniquely. Since $\mathrm{NR} = 0$ at point B, from (1.43) $Q = \mathrm{TDR}/M$.

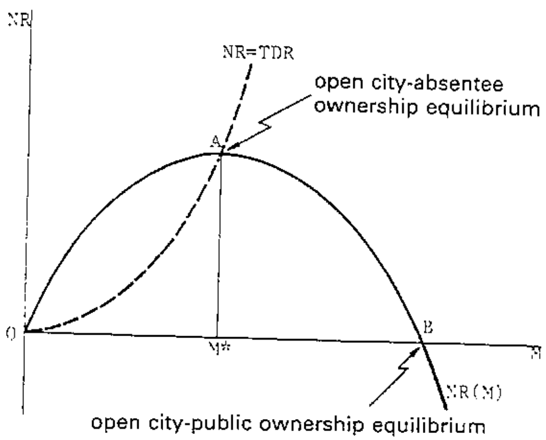

FIGURE 1.9 Relationship between optimal solutions and equilibrium solutions (u fixed).

Utilizing this result, it is not difficult to confirm that if M is given at point B, optimality conditions (1.43)–(1.36) coincides with the equilibrium conditions for the public ownership model of (1.22) under free migration. Hence, we can conclude that *if the national utility level is lower than $\hat{u}$, there exists a unique equilibrium with positive population for the open city model under public land ownership* (*case* 4). Finally, point A in Figure 1.9 corresponds to the optimal population for the city developer who aims to maximize the net revenue. At this point, $\partial\mathrm{NR}/\partial M = 0$. Applying the envelope theorem to (1.43), we have $\partial\mathrm{NR}/\partial M = -Q$. Hence, at point A, $\mathrm{NR} = \mathrm{TDR}$ and $Q = 0$. Utilizing this result, it is not difficult to confirm that point A corresponds to the equilibrium of the open city model under absentee land ownership (case 2).

1.4. Some extensions

Having mastered the basic model (1.1), it is appropriate in this subsection to incorporate some of the important factors which we have previously neglected.

A. Time cost of commuting. Although we have not explicitly considered the time cost of commuting, it is so important in practice that some authors (e.g., Beckmann [22]; Henderson [61], Hochman and Ofek [64]) considered only that cost, neglecting the pecuniary cost. In order to examine the effects of pecuniary cost and time cost on residential choice, let us consider the following model of residential choice, which is a simplified version of Yamada [155][25]

$$\max_{r,z,s,t_l,t_w} U(z, s, t_l), \quad \text{subject to} \tag{1.44}$$

$$z + R(r)s + ar = Y_N + wt_w, \quad \text{and} \quad t_l + t_w + br = \bar{t},$$

where z, s, and $R(r)$ are the same as before: t_l represents the leisure time, t_w the time for working, b the commuting time per distance, $\bar{t}$ the total available time, Y_N the nonwage income, w the wage rate, and a the pecuniary commuting cost per distance. That is, the

[25] In Yamada [155], other factors such as disutility of working time and commuting time and environmental external effects are also considered. Note that here the household can freely choose the length of working time. For the case where maximum working length is constrained, see Yamada [155] and Moses [92].